Phenomenological description of sedimentation in turbulent vortex tangles

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The aim of this Brief Report is to provide a simple intuitive derivation of the results for sedimentation velocity of a small spherical particle in a counterflow vortex tangle in turbulent superfluid. When the velocity of the tangle vortex lines is small as compared to that of the particle, our results reduce to those obtained previously by other authors through more complex arguments, except for a logarithmic dependence of one of the coefficients on the vortex line density. Comparison of both derivations may be useful to clarify the range of validity of the expressions for the forces between the particle and the tangle.

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I. INTRODUCTION

The interest in the motion of small particles—micron size, as a rule—in turbulent vortex tangles has dramatically increased in the past two years because of the perspectives of applying particle image velocimetry to obtain new experimental information on superfluid turbulence in ^{4}He.[1–6] In order to understand and interpret the information obtained in this way, it is essential to know in detail the interaction of the particles with the vortex lines forming the tangle, which is a complicated problem. Thus, the particles will not simply be carried along with the normal component, but their motion will be modified by the vortex lines. The final aim of the analyses would be to get information on the normal fluid velocity and the vortex tangle structure. The final achievement of this aim will only be possible from a detailed microscopic understanding of several aspects of these interactions. However, in the meantime, phenomenological descriptions may also be helpful in focusing the attention on a systematic view of observable phenomena. Among these phenomenological descriptions, one could have the study of the diffusion of particles in turbulent tangles, the fluctuations of the position of one particle, or the effects of particles on some aspects of turbulence, as, for instance, their role as seeding particles lowering the intensity of turbulent fluctuations, as done in turbulence of classical fluids.

II. HYDRODYNAMICAL DESCRIPTION OF SEDIMENTATION

Here, we propose a hydrodynamical description of the sedimentation of small particles in a vortex tangle, which goes beyond previous hydrodynamical descriptions of inhomogeneous vortex tangles.[7–12]

To do so, we determine the force that the vortex tangle is expected to exert on the particles. In fact, when we refer to sedimentation, we do not necessarily mean the descending motion of the particles under the action of the gravitational force, because the particles are dragged by the ascending motion of the normal fluid. To be specific, we will consider a counterflow vortex tangle in a container heated from below. The heat so communicated is carried away from the lower plate by the normal (viscous) fluid with a speed $\mathbf{v}_n=\mathbf{q}/\rho sT$, with $\mathbf{q}$ being the heat flux, ρ being the mass density, and s being the entropy. In the so-called counterflow situations, the ascending flow of the normal component is compensated by a descending flow of the superfluid component, in such a way that the barycentric speed $\mathbf{v}$ (defined as $\rho\mathbf{v}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n$, with ρ_s, ρ_n, $\mathbf{v}_s$, and $\mathbf{v}_n$ as the densities and velocities of the normal and superfluid components) is 0, i.e., $\mathbf{v}_s=-(\rho_n/\rho_s)\mathbf{v}_n$. When $\mathbf{v}_n-\mathbf{v}_s=\mathbf{V}_{ns}$ exceeds a critical value, a vortex tangle is formed with vortex line density L. The relation between L and $\mathbf{V}$ may be obtained from suitable evolution equations for L.[13–16]

The particle is assumed to have a radius a and a mass density ρ_p. If the particle was submitted to the viscous fluid only, the force exerted on it by the fluid would be given by the well known Stokes law, namely,

$$\mathbf{F}_{\text{Stokes}}=-6\pi\eta a(\mathbf{v}_p-\mathbf{v}_n), \quad (1)$$

where η is the shear viscosity and $\mathbf{v}_p$ is the velocity of the particle. The terminal velocity of the sphere would then be given by the following classical relation:

$$-6\pi a\eta(\mathbf{v}_n-\mathbf{v}_p)=(4\pi a^3/3)(\rho_p-\rho)\mathbf{g}, \quad (2)$$

with $\rho=\rho_s+\rho_n$ as the density of the fluid. The acceleration of gravity $\mathbf{g}$ points downward and $\mathbf{v}_n$ is supposedly upward.

From a phenomenological point of view, one would search for the effects of the vortex lines on the particle in the form of forces related to the velocity between the particle and the tangle. In fact, it will be seen that this ansatz is not fully realized, but we take it as a provisional starting point, and we obtain the suitable corrections after a detailed macroscopic analysis of the phenomenology. In contrast with previous analyses, which have worked out a more microscopic and complex way, our aim here is to contribute to this problem with a simpler, more macroscopic, and intuitive presentation.

As the particle moves across the tangle, it is expected to drag vortex loops, break vortex lines, connect briefly to them, and stick to them (Fig. 1). Here, we will consider two situations: (1) the particle has one or several vortex loops attached to it and moving with it, or (2) the sphere moves across the tangle connecting to vortex lines, dragging them

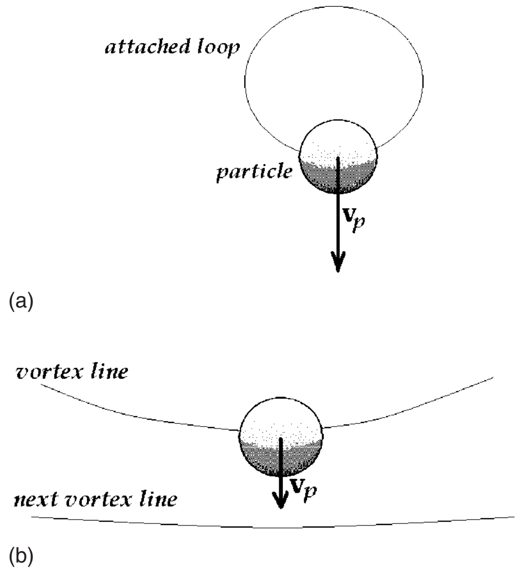

FIG. 1. The two situations we have considered are the following: (a) the particle in motion drags a vortex loop attached to it and moving with it, and (b) the particle connects to a vortex line and elongates it until it arrives at the next vortex line, where it connects from the former line to the latter.

until the next vortex line, and commuting then from a vortex line to the next one. In fact, these two processes are not necessarily additive. After discussing them separately, we will discuss up to what point they could interfere with each other.

A. Vortex loops attached to the particle and moving with it

Assume that the total length of the vortex loops, which are attached to a particle and dragged by it, is l. The force experienced by a vortex line of unit length moving across the normal component is $\alpha\rho_s\kappa(\mathbf{v}_p-\mathbf{v}_n)$, where we assume that the vortex loops move with the same speed as the particle. Here, $\kappa=h/m$ is the quantum of vorticity (h is the Planck constant and m is the atomic mass of helium) and α is a numerical coefficient. Thus, the drag force will be

$$\mathbf{F}_{\text{loops}} = -\gamma_0 l(\mathbf{v}_p-\mathbf{v}_n) \approx -\beta_1\gamma_0 a(\mathbf{v}_p-\mathbf{v}_n), \tag{3}$$

where we have assumed that $l=\beta_1 a$, with β_1 being a numerical coefficient of order unity. Note that this expression is similar to the Stokes one, with $\alpha\rho_s\kappa=\eta_{\text{eff}}$ playing the role of an effective viscosity as, indeed, this has dimensions of dynamical viscosity. To propose Eq. (3), we have taken into account that the mutual friction force between the unit length of the vortex lines and the normal fluid can be written[13] as follows:

$$\mathbf{f}_{MF} = -\gamma_0\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{v}_n-\mathbf{v}_L)]+\gamma_0'\mathbf{s}'\times(\mathbf{v}_n-\mathbf{v}_L), \tag{4}$$

with $\mathbf{v}_L$ being the velocity of the vortex line element, $\gamma_0=\rho_s\kappa\alpha/[(1-\alpha')^2+\alpha^2]$, and $\gamma_0'=\rho_s\kappa[\alpha^2-\alpha'(1-\alpha')]/[(1-\alpha')^2+\alpha^2]$, and we have considered in l the projection of the total length of the dragged loops orthogonal to the velocity of the sphere. We have neglected the transverse force [the second term in Eq. (4)] because it does not contribute to the drag of the particle, but simply moves the loop on the particle. In fact, here we are not referring to all vortex lines, but only to the vortex line constituting the loop attached to the particle, which is assumed to move with the velocity of the particle $\mathbf{v}_p$; thus, the velocity difference in Eq. (3) does, indeed, correspond to $\mathbf{v}_p-\mathbf{v}_n$.

B. Vortex lines elongated by the particle briefly attached to them

The force exerted by a vortex line, which is elongated by the motion of a particle connected to it, is of the order of ϵ_V, the energy per unit length of a vortex line. The coefficient ϵ_V is given by $\epsilon_V=(\rho_s\kappa^2/4\pi)\ln(L^{-1/2}/\xi)$, with ξ of the order of the vortex core radius (some 0.1 nm).[13,14] Now, we will consider the total force on a particle moving across the vortex tangle. To do so, we take into account that the number of vortex lines broken down per unit time by a particle of radius a moving at a speed $\mathbf{v}_p$ across a tangle of line density L is $2\pi a|\mathbf{v}_p-\mathbf{v}_L|L$. Indeed, $2\pi a|\mathbf{v}_p-\mathbf{v}_L|$ is the lateral area of the cylinder of radius a, which is described by the motion of the sphere per unit time; this area is crossed—in homogeneous turbulence—by L crossings per unit area. Now, to obtain the time-averaged force, we write

$$\mathbf{F}_{\text{tangle}} = -(\epsilon_V\Delta t)2\pi a L(\mathbf{v}_p-\mathbf{v}_L), \tag{5}$$

where Δt is the average time that the particle spends in every connection to a vortex line.

In the case we are studying, we must distinguish two situations: a dilute case, in which the average separation between vortices $L^{-1/2}$ is much higher than a, namely, $L^{-1/2}\gg a$; or a dense situation, in which the radius a is higher than $L^{-1/2}$. In the first case, the time interval between successive vortex lines is of the order of $\Delta t \simeq \beta_2 L^{-1/2}/|\mathbf{v}_p-\mathbf{v}_L|$, with β_2 a numerical constant of order 1. In the case of a dense situation, the particle is always connected to vortex lines, and the average time it is attached to one of them is $\Delta t\simeq\beta_2 a/|\mathbf{v}_p-\mathbf{v}_L|$. By introducing these expressions for Δt in Eq. (5), one finds for the force acting on the vortex line the following:

$$\mathbf{F}_{\text{tangle}}(\text{dilute}) = -\frac{\rho_s\kappa^2}{4\pi}\ln\left(\frac{L^{-1/2}}{\xi}\right)\beta_2 2\pi a L^{1/2}\frac{\mathbf{v}_p-\mathbf{v}_L}{|\mathbf{v}_p-\mathbf{v}_L|} \tag{6}$$

and

$$\mathbf{F}_{\text{tangle}}(\text{dense}) = -\frac{\rho_s\kappa^2}{4\pi}\ln\left(\frac{L^{-1/2}}{\xi}\right)\beta_2 2\pi a^2 L\frac{\mathbf{v}_p-\mathbf{v}_L}{|\mathbf{v}_p-\mathbf{v}_L|}. \tag{7}$$

Thus, as we have anticipated, the initial ansatz of forces related to the velocity has led us, in Eqs. (6) and (7), to a final form of forces that do not depend on the modulus of the relative velocity, because it also appears in the denominator and it cancels out. Since Eqs. (6) and (7) do not depend on the modulus of $\mathbf{v}_p-\mathbf{v}_L$, they belong to the so-called dry friction force.[7,11,13]

In fact, still a third possibility would be to consider that the particle gets stuck to a vortex line and cannot fleet away. This may happen if the relative velocity of the particle is sufficiently small; thus, a description at very low velocities requires knowing the minimum value of the velocity necessary not to get stuck to a vortex line, but these aspects are not yet clearly settled and we will not consider them.

As we have commented, the two processes in Secs. II A and II B may interfere with each other in two ways. It could be that the vortex loop attached to the particle (Sec. II A) is lost during the connection-elongation-reconnection process (Sec. II B); inversely, it could be that in the reconnection part of the process, the particle keeps a vortex loop on it for some time. Therefore, the length of the attached vortex loop, which is related to the coefficient β_1 in Eq. (3), must not be imagined as a constant parameter, but as a time-averaged parameter along the sedimentation. Analogously, the connection-elongation-reconnection processes could be influenced by the loop attached to the particle in such a way that the coefficient β_2 in Eqs. (6) and (7) must also be thought of as a time-averaged quantity over the sedimentation process. If one is interested in the behavior of the sedimentation along a time much longer than the time scale of the connection-elongation-reconnection process, the time variations in the coefficients β_1 and β_2 should be smoothed out. In contrast, if one is interested in the short scale behavior of the particle, the fluctuating time behavior of these coefficients should be taken into account. Here, according to the available experimental data, we will only assume the long-time behavior of the sedimentation.

Now, we return to the sedimentation problem by recalling that the particle is dragged upward by the normal fluid and downward by the gravitation. The corresponding balance of forces at sedimentation velocity is as follows:

$$-[6\pi\eta+\beta_1\gamma_0]a(\mathbf{v}_n-\mathbf{v}_p)+\frac{\beta_2}{2}\rho_s\kappa^2\ln\left(\frac{L^{-1/2}}{\xi}\right)aL^{1/2}\frac{\mathbf{v}_p-\mathbf{v}_L}{|\mathbf{v}_p-\mathbf{v}_L|} = \frac{4\pi a^3}{3}(\rho_p-\rho)\mathbf{g}. \quad (8)$$

If we suppose that the velocities $\mathbf{v}_L$, $\mathbf{v}_p$, and $\mathbf{v}_n$ are collinear with $\mathbf{g}$, Eq. (8) yields

$$v_p=\frac{2a^2(\rho_p-\rho)}{9\eta\left[1+\frac{\beta_1}{6\pi}\gamma_0\right]}g+v_n\left[1-\frac{\beta_2\gamma_H}{2}\rho\kappa\ln\left(\frac{L^{-1/2}}{\xi}\right)a\right], \quad (9)$$

where we have taken into account that in sufficiently well-developed counterflow turbulence, one has $L=\gamma_H^2(V_{ns}/\kappa)^2$ (Refs. 13 and 14) and that $V_{ns}=v_n-v_s=(\rho/\rho_s)v_n$ according to the counterflow constraint $v_s=-(\rho_n/\rho_s)v_n$. In fact, this estimation could be slightly improved by taking into account that a better expression for $L^{1/2}$ is $L^{1/2}=\gamma_H(V_{ns}/\kappa)-(b_1/d)$, where d is the width of the channel and b_1 is a numerical constant.[15]

We now consider the case in which the particle density is equal to the helium density. In this situation, Eq. (8) yields

$$-[6\pi\eta+\beta_1\gamma_0](\mathbf{v}_n-\mathbf{v}_p)+\frac{\beta_2}{2}\rho_s\kappa^2L^{1/2}\ln\left(\frac{L^{-1/2}}{\xi}\right)\mathbf{u}_{pL}=0, \quad (10)$$

with $\mathbf{u}_{pL}$ as the unit vector in the direction of $\mathbf{v}_p-\mathbf{v}_L$, from which we deduce that the relative velocity $\mathbf{v}_n-\mathbf{v}_p$ has the direction of $\mathbf{v}_p-\mathbf{v}_L$, while its modulus is expressed as

$$|\mathbf{v}_n-\mathbf{v}_p|=\frac{\frac{\beta_2}{2}\rho_s\kappa^2L^{1/2}\ln\left(\frac{L^{-1/2}}{\xi}\right)}{6\pi\eta+\beta_1\gamma_0}=\frac{\frac{\beta_2}{2}\rho\kappa\gamma_H\ln\left(\frac{L^{-1/2}}{\xi}\right)}{6\pi\eta+\beta_1\gamma_0}v_n. \quad (11)$$

Therefore, in this situation, it is not always true that $\mathbf{v}_p$ and $\mathbf{v}_n$ are collinear. We also note that from Eqs. (8) and (11), we cannot determine the value of $\mathbf{v}_L$.

In the dense situation, one would have the following instead of Eq. (9):

$$v_p=\frac{2a^2(\rho_p-\rho)}{9\eta\left[1+\frac{\beta_1}{6\pi}\gamma_0\right]}g+v_n\left[1-\frac{\beta_2\gamma_H^2}{2}\rho\kappa\ln\left(\frac{L^{-1/2}}{\xi}\right)a^2\right]. \quad (12)$$

To compare Eqs. (9) and (12) with equations in Ref. 3, recall that $\mathbf{g}=-g\hat{\mathbf{z}}$, $\mathbf{v}_p=-v_p\hat{\mathbf{z}}$, and $\mathbf{v}_n=v_n\hat{\mathbf{z}}$, with $\hat{\mathbf{z}}$ being the unit vector in the upward direction. Equations (9) and (12) are analogous to those obtained in Ref. 3, but with the difference that in the logarithmic contribution to the second term, we have $\ln(L^{-1/2}/\xi)$, whereas those authors have $\ln(a/\xi)$. In the dense tangle, we have considered that $L^{-1/2}\approx a$ and, therefore, the results are identical. In the dilute situation, the results differ logarithmically. The numerator and the denominator in the logarithmic factor in Eqs. (6) and (7) represent the radial cutoff in the integration of the velocity profile of the vortex, which has the form $\kappa/2\pi r$. In a pure tangle, the integration is carried out from the core of the vortex, ξ, to the next neighbor vortex line, which is at an average distance $L^{1/2}$. In the case of the vortex attached to a sphere of radius a, the radius a seems, in principle, a reasonable choice, as made in Ref. 3. However, in a more detailed consideration of the situation, this is not so obvious, as the vortices are orthogonal to the sphere and, therefore, the radial velocity profile should be extended to the neighbor vortex line instead of the radius of the sphere, which is in contrast to the assumption made in Ref. 3.

III. CONCLUDING REMARKS

The last term in Eqs. (9) and (12) has received much attention since the experimental findings by Zhang and Van Sciver[1] and has been theoretically derived by Sergeev *et al.*[3] The terms in β_2 are of special interest because they would not appear in a naive hydrodynamic derivation of the expression for the sedimentation velocity. If the density ρ_p of the particle is taken equal to the total density of the superfluid, the first term in Eqs. (9) and (12) would vanish. Even in this situation, the velocity $\mathbf{v}_p$ of the particle would not be equal to the velocity $\mathbf{v}_n$ of the normal component. This important observation was made in Refs. 1–6.

Our derivation of these terms differs from theirs in terms of the numerator of the logarithmic factor and in the presence of the velocity v_L of the vortex lines in Eq. (8), which we have discussed above, and in the fact that it is more hydrodynamically minded, which is in contrast with Ref. 3 that starts from a more detailed view of vortex-particle relations.

Another piece of information that would be obtained from suspended particles would be their positional fluctuations, which still require much analysis. Here, we point out one of the aspects of these fluctuations, which is related to the vortex fluctuations. In a previous paper, we estimated the fluctuations in the length of vortex loops[11] and here we suggest in a speculative way to relate them with the fluctuations of the position of the suspended particles, $\delta\mathbf{r}$. If a particle was closely attached to a vortex line, the fluctuations in its position would be proportional to the fluctuation of the radius of the vortex loop, which are proportional to the fluctuation of the vortex length, whereas if the particle is far from the vortex, it will not feel this effect. Then, we tentatively write

$$\delta\mathbf{r} \sim \frac{a}{d}\frac{\delta\mathcal{L}}{N}, \tag{13}$$

with $d \approx L^{-1/2}$ as the average separation between vortices, $\mathcal{L} \equiv LV$ as the total vortex length in the volume V, and $N = \mathcal{L}/\langle l \rangle$ as the average number of vortices, with $\langle l \rangle$ being the average vortex loop length. Then we have the following:

$$\langle(\delta\mathbf{r})^2\rangle_{\text{diluted}} \sim a^2 L \frac{1}{N^2}\langle(\delta\mathcal{L})^2\rangle = a^2\frac{\langle l \rangle^3}{V}, \tag{14}$$

where we have taken our result $\langle(\delta L)^2\rangle = \langle l \rangle L$.[11] This result would be valid in the dilute regime. In the dense vortex regime, the factor a/d in Eq. (13) should be set equal to 1, and one would have the following:

$$\langle(\delta\mathbf{r})^2\rangle_{\text{dense}} \sim \frac{\langle l \rangle^3}{LV}. \tag{15}$$

This is an estimation of the influence of the vortex fluctuations on particle fluctuations, but a rigorous and detailed understanding of this problem is far from being achieved. Of course, this contribution should be added to the usual expression for the position fluctuations coming from Brownian motion.

However, to ensure that the particle will follow the fluctuations of the vortex, it is necessary that the particle is kept attached to it, but the trapping probability has not yet been calculated.[6] If it is not attached, the particle will separate rapidly from the vortex, as discussed in Ref. 6, with the characteristic time of separation being given by a^2/κ. For particles of the order of $a = 10^{-4}$ cm, which is the typical size of particles used in particle image velocimetry, the time of separation is of the order of 10^{-5} s, which is a very short time. Thus, it seems that unless the trapping probability is not considerably high, the fluctuations of the vortices will decouple from those of the particle.

Let us close our paper by suggesting the interest of exploring polarized and partially polarized tangles, as those arising in the simultaneous presence of rotation and counterflow.[16] The model proposed here basically refers to the interaction between the vertical motion of the particles and the horizontal vortex lines. In rotating systems, rotation orients the vortex lines parallel to the rotation axis and the tangle is not isotropic. Thus, the sedimentation speed would be expected to depend on the angular rotation of the container. By analyzing this situation, a deeper exploration of other aspects of particle-vortex interaction could be stimulated.

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